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DOI:
10.1016/j.jde.2018.04.051

Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 265 (2018) 3345–3362
Journal of Differential Equations
www.elsevier.com/locate/jde

On a L^∞ functional derivative estimate relating to the Cauchy problem for scalar semi-linear parabolic partial differential equations with general continuous nonlinearity

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Received 22 September 2017; revised 16 February 2018
Available online 14 June 2018

Abstract

In this paper, we consider a L^∞ functional derivative estimate for the first spatial derivatives of bounded classical solutions $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ to the Cauchy problem for scalar second order semi-linear parabolic partial differential equations with a continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ and initial data $u_0 : \mathbb{R}^N \to \mathbb{R}$, of the form,

$$\max_{i=1,\dots,N}\left(\sup_{x\in\mathbb{R}^N} |u_{x_i}(x,t)|\right) \le \mathcal{F}_t(f, u_0, u) \quad \forall t \in [0, T].$$

Here $\mathcal{F}_t : \mathcal{A}_t \to \mathbb{R}$ is a functional as defined in §1 and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^N$. We establish that the functional derivative estimate is non-trivially sharp, by constructing a sequence $(f_n, 0, u^{(n)})$, where for each $n \in \mathbb{N}$, $u^{(n)} : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ is a solution to the Cauchy problem with zero initial data and nonlinearity $f_n : \mathbb{R} \to \mathbb{R}$, and for which there exists $\alpha > 0$ such that

$$\max_{i=1,\dots,N}\left(\sup_{x\in\mathbb{R}^N} |u^{(n)}_{x_i}(x,T)|\right) \ge \alpha,$$

whilst

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https://doi.org/10.1016/j.jde.2018.04.051

$$\lim_{n\to\infty}\left(\inf_{t\in[0,T]}\left(\max_{i=1,\dots,N}\left(\sup_{x\in\mathbb{R}^N}|u^{(n)}_{x_i}(x,t)|\right)-\mathcal{F}_t(f_n,0,u^{(n)})\right)\right)=0.$$

MSC: 35K58; 26D10; 35B45

Keywords: Semi-linear parabolic PDE; Functional derivative estimate

1. Introduction

In this paper we introduce and consider the sharpness of a functional derivative estimate (see Proposition 1.1) for solutions $u:\bar{D}_T\to\mathbb{R}$ to the Cauchy problem for the scalar second order semi-linear parabolic partial differential equation ($T>0$) given by,

$$u_t-\Delta u=f(u) \text{ on } D_T, \tag{1}$$

$$u=u_0 \text{ on } \partial D, \tag{2}$$

with $D_T=\mathbb{R}^N\times(0,T]$, $\partial D=\mathbb{R}^N\times\{0\}$, nonlinearity $f\in C(\mathbb{R})$ and initial data $u_0\in \mathrm{BPC}^1(\mathbb{R}^N)$, where $\mathrm{BPC}^1(\mathbb{R}^N)$ is the set of bounded, continuous real-valued functions defined on $\mathbb{R}^N$ which have piecewise continuous bounded gradient. We consider bounded solutions to the Cauchy problem (1)–(2) (henceforth referred to as [CP]), which are classical, in the sense that

$$u\in C(\bar{D}_T)\cap C^{2,1}(D_T)\cap L^\infty(\bar{D}_T). \tag{3}$$

Related to [CP], for any given $T>0$ we introduce the sets $\mathcal{A}_T$ and $\mathcal{I}_T$ given by

$$\mathcal{A}_T=\{(f,v,u):\ f\in C(\mathbb{R}),\ v\in \mathrm{BPC}^1(\mathbb{R}^N),\ u\in C(\bar{D}_T)\cap L^\infty(\bar{D}_T)\},$$

and

$$\begin{aligned}\mathcal{I}_T=\{(f,u_0,u):\ &(f,u_0,u)\in\mathcal{A}_T \text{ and}\\ &u:\bar{D}_T\to\mathbb{R} \text{ is a solution to [CP] with } f \text{ and } u_0\}.\end{aligned} \tag{4}$$

For any $T>0$, we observe trivially that $\mathcal{I}_T\subset\mathcal{A}_T$, and $\mathcal{I}_T$ is non-empty ($(f,u_0,u)\in\mathcal{I}_T$, with each of f, u_0 and u being the zero function). In addition, when $f\in H_\alpha$ (functions which are Hölder continuous of degree $0<\alpha\le 1$ on every closed bounded interval $E\subset\mathbb{R}$) and $u_0\in \mathrm{BPC}^1(\mathbb{R}^N)$, with the corresponding [CP] being *a priori* bounded on $\bar{D}_T$, then [CP] has a solution $u:\bar{D}_T\to\mathbb{R}$ (see, for example, [1], [2], [3]), and $(f,u_0,u)\in\mathcal{I}_T$. For any $T>0$, we also note that when $f=f_p:\mathbb{R}\to\mathbb{R}$ (for any $0<p<1$) is given by

$$f_p(u)=u|u|^{p-1}\quad \forall u\in\mathbb{R} \tag{5}$$

and $u_0 : \mathbb{R}^N \to \mathbb{R}$ is given by $u_0(x) = 0$ for all $x \in \mathbb{R}^N$, it has been established in [4] that there exists non-trivial $u = u_p : \bar{D}_T \to \mathbb{R}$ such that $(f_p, 0, u_p) \in \mathcal{I}_T$. We will examine [CP] with $f = f_p$ in detail, in §2 and §4 to establish our main result, Theorem 1.3, namely that the derivative estimate in Proposition 1.1 is *non-trivially sharp*.

Optimal Schauder estimates for solutions to Cauchy problems for second order semi-linear parabolic partial differential equations, such as (1)–(3), but also with more general linear terms in (1), have been developed in [5], [6], [7] and [8]. A review of these developments is contained in [9]. In these works, the nonlinear terms in the partial differential equations (as well as the coefficients of the linear terms) are locally Hölder continuous and optimality is meant in the sense of the best possible regularity of solutions, i.e. the most refined Hölder space of functions in which solutions are contained. Specifically these Hölder spaces consist of functions where the first time derivative and first and second spatial derivatives exist and are Hölder continuous to a degree dependent on the Hölder degree of the nonlinear terms and linear coefficients in the partial differential equation. Notably, optimality in this paper is not meant in the sense of optimal generic derivative bounds when, say, the Hölder degree of the nonlinearity is specified. Consequently, for [CP], the optimal derivative estimates for first spatial derivatives of solutions to [CP] contained in [5], [6], [7], [8] and [9] (corresponding to the estimate in Corollary 1.2) are curiously larger than those which the functional derivative estimate in Proposition 1.1 yields; this has motivated our introduction of the notion of a derivative estimate being *non-trivially sharp*.

The paper is structured as follows. In the remainder of §1, we establish a Schauder-type derivative estimate and a functional derivative estimate for the first spatial derivatives of solutions to [CP]. In addition, we motivate and define the notion of the functional derivative estimate being *non-trivially sharp* and state the main result of the paper in Theorem 1.3. In §2, for fixed $T > 0$ and each $0 < p < 1$, we introduce $(f_p, 0, u^{(p)}) \in \mathcal{I}_T$ with $u^{(p)} : \bar{D}_T \to \mathbb{R}$ a specific non-trivial anti-symmetric self similar solution to [CP], which corresponds to the front solution in [4]. In §3, we consider the formal limit as $p \to 0$ of a boundary value problem for the ordinary differential equation associated with $u^{(p)}$. This is used in §4 to establish Theorem 1.3. Finally in §5 we discuss alternative approaches to establishing Theorem 1.3, as well as related concluding remarks.

Before we state our main result, we introduce notation and establish preliminary results as a necessity, but also as motivation. To begin, for $u \in L^\infty(\bar{D}_T)$ and $\lambda \in (0, T]$, we denote $||u||_\infty^\lambda = ||u|_{\bar{D}_\lambda}||_\infty$. Also, for $\lambda > 0$, we introduce the functional $F_\lambda : \mathcal{A}_\lambda \to [0, \infty)$ given by

$$\mathcal{F}_\lambda(f, v, u) = \max_{i=1,\dots,N} ||v_{x_i}||_\infty + \frac{1}{\sqrt{\pi}} \int_0^\lambda \frac{||f(u(\cdot, \tau))||_\infty}{(\lambda - \tau)^{1/2}} d\tau \quad \forall (f, v, u) \in \mathcal{A}_\lambda. \tag{6}$$

It follows immediately from (6) that for each $\lambda > 0$, the functional $\mathcal{F}_\lambda : \mathcal{A}_\lambda \to [0, \infty)$ is well-defined, and, for each $(f, v, u) \in \mathcal{A}_\lambda$, the following inequality holds

$$\mathcal{F}_\lambda(f, v, u) \le \max_{i=1,\dots,N} ||v_{x_i}||_\infty + \frac{2\lambda^{1/2}}{\sqrt{\pi}} ||f(u)||_\infty^\lambda. \tag{7}$$

We now consider a functional derivative estimate for [CP], which is a straightforward extension of those given in [10, Lemma 5.12] and [11, Lemma 3.9].

Proposition 1.1 *(Functional derivative estimate). Let $(f, u_0, u) \in \mathcal{I}_T$. Then, for each $0 < t \le T$, it follows that $(f, u_0, u|_{\bar{D}_t}) \in \mathcal{I}_t$, $u_{x_i}(\cdot, t) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ $(i = 1, \dots, N)$ and*

$$\max_{i=1,\dots,N} ||u_{x_i}(\cdot,t)||_\infty \leq \mathcal{F}_t(f,u_0,u|_{\bar{D}_t}).$$

Proof. Since $u : \bar{D}_T \to \mathbb{R}$ is a solution to [CP] with f and u_0, it follows by definition that $u|_{\bar{D}_t}$ is a solution to [CP] with f and u_0 on $\bar{D}_t$ for any $0 < t \leq T$, and hence, $(f, u_0, u|_{\bar{D}_t}) \in \mathcal{I}_t$. For convenience we drop the restriction notation from here onward (with $(f, u_0, u) \in \mathcal{A}_T$, then $(f, u_0, u) \in \mathcal{A}_t$ for each $0 < t \leq T$). Now, let $(f, u_0, u) \in \mathcal{I}_T$. Then, since $f(u) \in C(\bar{D}_T) \cap L^\infty(\bar{D}_T)$, it follows from a standard application of the finite Laplace transform to (1) and (2), via (3) (see, for example [10, Theorem 4.9] with $N = 1$), or from the variation-of-constants formula, that $u : \bar{D}_T \to \mathbb{R}$ satisfies the following integral equation,

$$\begin{aligned} u(x,t) =& \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} u_0(x + 2\sqrt{t}w) e^{-|w|^2} dw \\ &+ \frac{1}{\pi^{N/2}} \int_0^t \int_{\mathbb{R}^N} f(u(x + 2\sqrt{t-\tau}w, \tau)) e^{-|w|^2} dw d\tau \quad \forall (x,t) \in D_T. \end{aligned} \tag{8}$$

Again, since $f(u) \in C(\bar{D}_T) \cap L^\infty(\bar{D}_T)$ and $u_0 \in \mathrm{BPC}^1(\mathbb{R}^N)$, we observe that both terms on the right hand side of (8) have continuous partial derivatives with respect to x_j on D_T for $j = 1, \dots, N$ (see, for example, [10, Lemma 5.9] with $N = 1$) and it follows that for each $j = 1, \dots, N$,

$$\begin{aligned} u_{x_j}(x,t) =& \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} u_{0x_j}(x + 2\sqrt{t}w) e^{-|w|^2} dw \\ &+ \frac{1}{\pi^{N/2}} \int_0^t \int_{\mathbb{R}^N} \frac{f(u(x + 2\sqrt{t-\tau}w, \tau))}{(t-\tau)^{1/2}} w_j e^{-|w|^2} dw d\tau \quad \forall (x,t) \in D_T. \end{aligned} \tag{9}$$

Therefore, for each $j = 1, \dots, N$,

$$\begin{aligned} |u_{x_j}(x,t)| &\leq ||u_{0x_j}||_\infty + \frac{1}{\pi^{N/2}} \int_0^t \int_{\mathbb{R}^N} \left| \frac{f(u(x + 2\sqrt{t-\tau}w, \tau))}{(t-\tau)^{1/2}} w_j e^{-|w|^2} \right| dw d\tau \\ &\leq ||u_{0x_j}||_\infty + \frac{1}{\sqrt{\pi}} \int_0^t \int_{\mathbb{R}} \frac{||f(u(\cdot,\tau))||_\infty}{(t-\tau)^{1/2}} |w_j| e^{-w_j^2} dw_j d\tau \qquad (10) \\ &\leq \max_{j=1,\dots,N} ||u_{0x_j}||_\infty + \frac{1}{\sqrt{\pi}} \int_0^t \frac{||f(u(\cdot,\tau))||_\infty}{(t-\tau)^{1/2}} d\tau \\ &= \mathcal{F}_t(f, u_0, u) \quad \forall (x,t) \in D_T. \qquad (11) \end{aligned}$$

Since the right hand side of (11) is independent of $x \in \mathbb{R}^N$ and $(f, u_0, u) \in \mathcal{I}_T$, it is an immediate consequence that for each $0 < t \leq T$, $u_{x_i}(\cdot, t) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ $(i = 1, \dots, N)$, and the required inequality follows trivially. □

A derivative estimate can now be obtained as follows,

Corollary 1.2. *Let* $(f, u_0, u) \in \mathcal{I}_T$. *Then,*

$$\max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty \leq \max_{i=1,\dots,N} ||u_{0x_i}||_\infty + \frac{2t^{1/2}}{\sqrt{\pi}} ||f(u)||_\infty^t \quad \forall t \in (0, T].$$

Proof. This follows directly from Proposition 1.1 and (7). □

For each $(f, u_0, u) \in \mathcal{I}_T$, we now have, via Proposition 1.1 and (7), that

$$-\left(\max_{i=1,\dots,N} ||u_{0x_i}||_\infty + \frac{2T^{1/2}}{\sqrt{\pi}} ||f(u)||_\infty^T\right) \leq \max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f, u_0, u) \leq 0 \quad \forall t \in (0, T]. \tag{12}$$

Therefore, given $(f, u_0, u) \in \mathcal{I}_T$, then $(\max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f, u_0, u))$ is bounded uniformly above and below for $t \in [0, T]$. Moreover, via (12), it follows that for any $(f, u_0, u) \in \mathcal{I}_T$,

$$\begin{aligned} -\left(\max_{i=1,\dots,N} ||u_{0x_i}||_\infty + \frac{2T^{1/2}}{\sqrt{\pi}} ||f(u)||_\infty^T\right) &\leq \inf_{t \in (0,T]} \left(\max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f, u_0, u)\right) \\ &\leq \sup_{t \in (0,T]} \left(\max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f, u_0, u)\right) \\ &\leq 0. \end{aligned} \tag{13}$$

Now, motivated by (13), we refer to the derivative estimate in Proposition 1.1 as *sharp* on $\bar{D}_T$, when

$$\sup_{(f, u_0, u) \in \mathcal{I}_T} \left(\inf_{t \in (0,T]} \left(\max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f, u_0, u)\right)\right) = 0.$$

However, we observe that this definition is not immediately satisfactory; consider the case when $f^* \in C(\mathbb{R})$, $u_0^* \in \mathrm{BPC}^1(\mathbb{R}^N)$ and $u^* \in C(\bar{D}_T) \cap L^\infty(\bar{D}_T)$ are given by

$$f^*(u) = 0 \quad \forall u \in \mathbb{R}, \tag{14}$$

$$u_0^*(x) = 0 \quad \forall x \in \mathbb{R}^N, \tag{15}$$

$$u^*(x, t) = 0 \quad \forall (x, t) \in \bar{D}_T. \tag{16}$$

Then, trivially, $(f^*, u_0^*, u^*) \in \mathcal{I}_T$, with, for each $i = 1, \dots, N$,

$$||u_{x_i}^*(\cdot, t)||_\infty = 0 \quad \forall t \in (0, T], \tag{17}$$

whilst

$$\mathcal{F}_t(f^*, u_0^*, u^*) = 0 \quad \forall t \in (0, T]. \tag{18}$$

Thus, it follows from (17) and (18) that

$$\inf_{t\in(0,T]} \left(\max_{i=1,\dots,N} ||u^*_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f^*, u_0^*, u^*) \right) = 0, \tag{19}$$

and then from (19) and (13) that

$$\sup_{(f,u_0,u)\in\mathcal{I}_T} \left(\inf_{t\in(0,T]} \left(\max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f, u_0, u) \right) \right) = 0,$$

and hence, the derivative estimate in Proposition 1.1, according to the definition introduced above, is *sharp*. To exclude such triviality associated with spatially homogeneous solutions to [CP], we introduce the following refinement to the above definition; namely, we refer to the functional derivative estimate in Proposition 1.1 as *non-trivially sharp with index* α on $\bar{D}_T$ when there exists $\alpha > 0$ such that

$$\sup_{(f,u_0,u)\in\mathcal{I}^\alpha_T} \left(\inf_{t\in(0,T]} \left(\max_{i=1,\dots,N} ||u_{x_i}(\cdot, t)||_\infty - \mathcal{F}_t(f, u_0, u) \right) \right) = 0,$$

where now

$$I^\alpha_T = \{(f, u_0, u) : \ (f, u_0, u) \in \mathcal{I}_T \text{ and } \max_{i=1,\dots,N} ||u_{x_i}(\cdot, T)||_\infty \geq \alpha\}.$$

We can now state the main result in this paper, as

Theorem 1.3. *For any* $T > 0$ *and* $\alpha > 0$*, there exists a sequence* $\{(f_n, 0, u_n) \in \mathcal{I}^\alpha_T\}_{n\in\mathbb{N}}$*, such that*

$$\lim_{n\to\infty} \left(\inf_{t\in(0,T]} \left(\max_{i=1,\dots,N} ||u_{nx_i}(\cdot, t)||_\infty - \mathcal{F}_t(f_n, 0, u_n) \right) \right) = 0.$$

It then follows immediately from Theorem 1.3 that

Corollary 1.4. *For any* $T > 0$ *and* $\alpha > 0$*, the derivative estimate in Proposition 1.1 is non-trivially sharp with index* α *on* $\bar{D}_T$*.*

2. The problem (P_p)

For any fixed $N \in \mathbb{N}$ and $p \in (0, 1)$, consider [CP] with nonlinearity $f_p : \mathbb{R} \to \mathbb{R}$ given by (5) and initial data $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $u_0 = 0$. Henceforth we will refer to this as (P_p). In [4, Theorem 3.14] it is demonstrated that, for any $T > 0$, there exists a self-similar solution $u^{(p)} : \bar{D}_T \to \mathbb{R}$ to (P_p) of the form

$$u^{(p)}(x,t) = \begin{cases} w_p(\eta(x,t))t^{1/(1-p)}, & (x,t) \in D_T \\ 0, & (x,t) \in \partial D \end{cases} \tag{20}$$

with $\eta(x,t) = x_1 t^{-1/2}$ for all $(x,t) \in D_T$, whilst $w_p : \mathbb{R} \to \mathbb{R}$ is such that $w_p \in C^2(\mathbb{R})$, and

$$w_p'' + \frac{1}{2}\eta w_p' + f_p(w_p) - \frac{1}{(1-p)} w_p = 0 \quad \forall \eta \in \mathbb{R}, \tag{21}$$

$$w_p(-\eta) = -w_p(\eta) \quad \forall \eta \in \mathbb{R}, \tag{22}$$

$$|w_p(\eta)| < (1-p)^{1/(1-p)} \quad \forall \eta \in \mathbb{R}, \tag{23}$$

$$w_p(\eta) \to \pm(1-p)^{1/(1-p)} \text{ as } \eta \to \pm\infty, \tag{24}$$

$$0 < w_p'(\eta) < \sup_{\eta\in\mathbb{R}} |w_p'(\eta)| = w_p'(0) \quad \forall \eta \in \mathbb{R}\backslash\{0\}, \tag{25}$$

$$w_p'(0) > \frac{(1-p)^{1/(1-p)}}{(1+p)^{1/2}}. \tag{26}$$

The function $w_p : \mathbb{R} \to \mathbb{R}$, for $p \in (0,1)$, will be used extensively throughout the rest of the paper. Now, for any $T > 0$, since $u^{(p)} : \bar{D}_T \to \mathbb{R}$ given by (20) is a solution to (P_p), we have that

$$(f_p, 0, u^{(p)}) \in \mathcal{I}_t \quad \forall t \in (0,T]. \tag{27}$$

In addition, it follows from (5), (20), (23) and (24), that,

$$\mathcal{F}_t(f_p, 0, u^{(p)}) = \frac{(1-p)^{p/(1-p)}\Gamma(1/(1-p))}{\Gamma((3-p)/2(1-p))} t^{(1+p)/2(1-p)} \quad \forall t \in (0,T], \tag{28}$$

whilst from (20) and (25),

$$\max_{i=1,\dots,N} ||u_{x_i}^{(p)}(\cdot,t)||_\infty = ||u_{x_1}^{(p)}(\cdot,t)||_\infty = w_p'(0) t^{(1+p)/2(1-p)} \quad \forall t \in (0,T]. \tag{29}$$

Therefore, via (28), (29) and Proposition 1.1,

$$\max_{i=1,\dots,N} ||u_{x_i}^{(p)}(\cdot,t)||_\infty - \mathcal{F}_t(f_p, 0, u^{(p)}) = (w_p'(0) - \phi(p)) t^{(1+p)/2(1-p)} \le 0 \quad \forall t \in (0,T] \tag{30}$$

where $\phi : (0,1) \to \mathbb{R}$ is given by

$$\phi(p) = \frac{(1-p)^{p/(1-p)}\Gamma(1/(1-p))}{\Gamma((3-p)/2(1-p))} \quad \forall p \in (0,1). \tag{31}$$

Furthermore, by substitution into (10)

$$|f_p(u^{(p)}(x,t))| < ||f_p(u^{(p)}(\cdot,t))||_\infty \quad \forall (x,t) \in D_T,$$

which follows from (20), (23) and (24), and proceeding with the proof of Proposition 1.1 we may conclude that the inequality in (30) is, in fact, strict. We also observe that,

$$\phi(p) > 0 \quad \forall p \in (0,1), \tag{32}$$

$$\phi(p) \to \frac{2}{\sqrt{\pi}} \text{ as } p \to 0^+. \tag{33}$$

In addition, it follows from (7), (5), (20) and (23), that,

$$\mathcal{F}_t(f_p, 0, u^{(p)}) \le \frac{2}{\sqrt{\pi}}(1-p)^{p/(1-p)} t^{(1+p)/2(1-p)} \quad \forall t \in (0, T]. \tag{34}$$

Thus, via (28) and (34), we have,

$$\phi(p) \le \frac{2}{\sqrt{\pi}}(1-p)^{p/(1-p)} < \frac{2}{\sqrt{\pi}} \quad \forall p \in (0,1). \tag{35}$$

Now, we conclude from the discussion following (30) that

$$\inf_{t\in(0,T]}\left(\max_{i=1,\ldots,N} ||u_{x_i}^{(p)}(\cdot,t)||_\infty - \mathcal{F}_t(f_p, 0, u^{(p)})\right) = (w_p'(0) - \phi(p))T^{(1+p)/2(1-p)} < 0. \tag{36}$$

We also observe from (20) and (26) that

$$\max_{i=1,\ldots,N} ||u_{x_i}^{(p)}(\cdot,T)||_\infty = w_p'(0)T^{(1+p)/2(1-p)} > \frac{(1-p)^{1/(1-p)}}{(1+p)^{1/2}} T^{(1+p)/2(1-p)}. \tag{37}$$

A proof of Theorem 1.3 will now follow, up to minor detail, if we are able to construct a sequence $\{p_n\}_{n\in\mathbb{N}}$, such that $p_n \to 0$ as $n \to \infty$, and

$$w_{p_n}'(0) \to \frac{2}{\sqrt{\pi}} \text{ as } n \to \infty.$$

It is the construction of such a sequence which we now address. However, before proceeding to this, it is worth noting from (36), (35) and (26), that at this stage, we have

$$\frac{(1-p)^{1/(1-p)}}{(1+p)^{1/2}} < w_p'(0) < \phi(p) < \frac{2}{\sqrt{\pi}} \quad \forall p \in (0,1). \tag{38}$$

We now proceed by examining the solution $w_0 : \bar{\mathbb{R}}^+ \to \mathbb{R}$ to a boundary value problem in which the ordinary differential equation is the formal limiting differential equation of that in (21), as $p \to 0^+$. We then show that there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$, such that $p_n \to 0$ as $n \to \infty$, whilst, for any $X > 0$,

$$w_{p_n} \to w_0 \text{ and } w_{p_n}' \to w_0' \text{ uniformly on } [0, X] \text{ as } n \to \infty. \tag{39}$$

The result then follows on observing that $w_0'(0) = 2/\sqrt{\pi}$.

3. The problem (S_0)

In this section, we consider the boundary value problem obtained by taking the formal limit as $p \to 0$ in the associated problem for the ordinary differential equation studied in [4] when $p \in (0, 1)$. We seek a function $w : [0, \infty) \to \mathbb{R}$ such that $w \in C^1([0, \infty)) \cap C^2((0, \infty))$ and

$$w'' + \frac{1}{2}\eta w' - w = -1 \quad \forall \eta > 0, \tag{40}$$

$$w(0) = 0, \quad w(\eta) \to 1 \text{ as } \eta \to \infty, \tag{41}$$

$$w(\eta) > 0 \quad \forall \eta > 0. \tag{42}$$

We refer to this linear inhomogeneous boundary value problem as (S_0). We observe that the coefficients in (40) are continuous functions of $\eta \in [0, \infty)$. Thus, the homogeneous part of (40) has two real-valued basis functions $w_1, w_2 \in C^1([0, \infty)) \cap C^2((0, \infty))$ and a particular integral $\bar{w} \in C^1([0, \infty)) \cap C^2((0, \infty))$ after which it is straightforward to establish that (S_0) has a unique solution given by $\omega_0 : [0, \infty) \to \mathbb{R}$ with

$$w_0(\eta) = 1 - \frac{4}{\sqrt{\pi}}(2 + \eta^2)I(\eta) \quad \forall \eta \in [0, \infty), \tag{43}$$

where

$$I(\eta) = \int_{\eta}^{\infty} \frac{e^{-s^2/4}}{(2 + s^2)^2} ds \quad \forall \eta \in [0, \infty) \tag{44}$$

and we note that $I(\eta)$ is monotone decreasing in $\eta \in [0, \infty)$ with $I(0) = \sqrt{\pi}/8$ (see [12, pp. 302, 7.4.11]) and $I(\eta)$ decays exponentially as $\eta \to \infty$. Finally, we observe from (43) and (44) that

$$w_0'(0) = \frac{2}{\sqrt{\pi}}. \tag{45}$$

In the following section, we proceed to construct the sequence of functions $w_{p_n} : \mathbb{R} \to \mathbb{R}$ for which (39) holds.

4. Proof of Theorem 1.3

In this section, we construct a sequence $\{p_n\}_{n \in \mathbb{N}}$ such that $p_n \in (0, 1)$ for all $n \in \mathbb{N}$, $p_n \to 0$ as $n \to \infty$ and, for any $X > 0$, $w_{p_n} : \mathbb{R} \to \mathbb{R}$ satisfies

$$w_{p_n} \to w_0, \;\; w'_{p_n} \to w_0' \text{ uniformly on } [0, X] \tag{46}$$

as $n \to \infty$, where $w_0 : [0, \infty) \to \mathbb{R}$ is the unique solution to (S_0), given by (43)–(44). We note that via (46) and (45), we have

$$w'_{p_n}(0) \to \frac{2}{\sqrt{\pi}} \text{ as } n \to \infty,$$

which is crucial to the proof of Theorem 1.3.

Throughout this section we consider $w_p : \mathbb{R} \to \mathbb{R}$ restricted to the domain $[0, \infty)$, so that $w_p = w_p : [0, \infty) \to \mathbb{R}$. To begin, we obtain uniform bounds on w_p, w_p' and w_p'' for $p \in (0, 1)$. We have first,

Proposition 4.1. *Consider $w_p : [0, \infty) \to \mathbb{R}$ with $p \in (0, 1)$. Then,*

$$0 \le w_p(\eta) < 1 \quad \forall \eta \ge 0,$$
$$0 < w_p'(\eta) < \frac{2}{\sqrt{\pi}} \quad \forall \eta \ge 0,$$

and

$$|w_p(\eta_1) - w_p(\eta_2)| \le \frac{2}{\sqrt{\pi}} |\eta_1 - \eta_2| \quad \forall \eta_1, \eta_2 \ge 0.$$

Proof. It follows from (23), (25) and (38) that for $p \in (0, 1)$

$$0 \le w_p(\eta) \le (1-p)^{1/(1-p)} < 1 \quad \forall \eta \ge 0,$$

and

$$0 < w_p'(\eta) \le w_p'(0) < \frac{2}{\sqrt{\pi}} \quad \forall \eta \ge 0. \tag{47}$$

Therefore, via the mean value theorem with (47), we have

$$|w_p(\eta_1) - w_p(\eta_2)| \le \sup_{\theta \in [0,\infty)} w_p'(\theta) |\eta_1 - \eta_2| \le \frac{2}{\sqrt{\pi}} |\eta_1 - \eta_2| \quad \forall \eta_1, \eta_2 \ge 0,$$

as required. □

Additionally, we have

Proposition 4.2. *Consider $w_p : [0, \infty) \to \mathbb{R}$ with $p \in (0, 1)$. Then, for any $X > 0$,*

$$|w_p'(\eta_1) - w_p'(\eta_2)| \le \left(\frac{X}{\sqrt{\pi}} + 2 \right) |\eta_1 - \eta_2| \quad \forall \eta_1, \eta_2 \in [0, X].$$

Proof. Via the mean value theorem,

$$|w_p'(\eta_1) - w_p'(\eta_2)| \le \sup_{\theta \in [0,X]} |w_p''(\theta)| |\eta_1 - \eta_2| \quad \forall \eta_1, \eta_2 \in [0, X]. \tag{48}$$

Now, from (21), (23) and Proposition 4.1,

$$|w_p''(\theta)| \le \left| \frac{\theta}{2} w_p'(\theta) \right| + |w_p(\theta)|^p + \left| \frac{1}{(1-p)} w_p(\theta) \right| \le \frac{X}{\sqrt{\pi}} + 2(1-p)^{p/(1-p)} \le \frac{X}{\sqrt{\pi}} + 2 \tag{49}$$

for all $\theta \in [0, X]$. The result then follows from (48) and (49). □

Before we can obtain a result for ω_p'' corresponding to Proposition 4.1 and Proposition 4.2, we need the following,

Proposition 4.3. *Consider* $w_p : [0, \infty) \to \mathbb{R}$ *with* $p \in (0, 1/2]$. *Then,*

$$w_p(\eta) \geq \begin{cases} \frac{1}{8\sqrt{2}}\eta, & 0 \leq \eta \leq \eta' \\ \frac{\eta'}{8\sqrt{2}}, & \eta > \eta' \end{cases}$$

where

$$\eta' = \sqrt{\pi}\left(\sqrt{1 + \frac{1}{4\sqrt{2\pi}}} - 1\right) < 1. \tag{50}$$

Proof. It follows from (21) that

$$\int_0^\eta w_p''(s)ds = \int_0^\eta \left(-\frac{1}{2}sw_p'(s) + \frac{1}{(1-p)}w_p(s) - (w_p(s))^p\right)ds \quad \forall \eta \in [0, \infty). \tag{51}$$

For $p \in (0, 1/2]$ we next observe that $H_p : [0, (1-p)^{1/(1-p)}] \to \mathbb{R}$ given by

$$H_p(x) = \frac{1}{(1-p)}x - x^p \quad \forall x \in [0, (1-p)^{1/(1-p)}] \tag{52}$$

satisfies

$$H_p(x) \geq -(1-p)^{p/(1-p)} \geq -1 \quad \forall x \in [0, (1-p)^{1/(1-p)}]. \tag{53}$$

Thus, it follows from Proposition 4.2, (23), (51), (52) and (53) that

$$w_p'(\eta) - w_p'(0) \geq \int_0^\eta \left(-\frac{1}{\sqrt{\pi}}s - 1\right)ds \geq -\frac{1}{2\sqrt{\pi}}\eta^2 - \eta \quad \forall \eta \in [0, \infty). \tag{54}$$

Now, from (26) we also have

$$w_p'(0) > \frac{(1-p)^{1/(1-p)}}{(1+p)^{1/2}} \geq \frac{1}{4\sqrt{2}} \tag{55}$$

for all $p \in (0, 1/2]$. Therefore, it follows from (54) and (55) that

$$w_p'(\eta) \geq w_p'(0) - \frac{1}{2\sqrt{\pi}}\eta^2 - \eta \geq \frac{1}{4\sqrt{2}} - \frac{1}{8\sqrt{2}} = \frac{1}{8\sqrt{2}} \quad \forall \eta \in [0, \eta'] \tag{56}$$

with η' given by (50). Now, it follows from (22) and (56) that

$$w_p(\eta) = \int_0^\eta w_p'(s)ds \geq \int_0^\eta \frac{1}{8\sqrt{2}}ds = \frac{1}{8\sqrt{2}}\eta \quad \forall \eta \in [0, \eta'].$$

Finally, via (25), we have

$$w_p(\eta) \geq w_p(\eta') \geq \frac{\eta'}{8\sqrt{2}} \quad \forall \eta \in (\eta', \infty),$$

as required. □

We now have,

Remark 4.4. For each $X > 0$, it follows from Proposition 4.1 and Proposition 4.2 that $\{w_p\}_{p\in(0,1)}$ and $\{w_p'\}_{p\in(0,1)}$ are uniformly bounded and equicontinuous on $[0, X]$.

We next define the sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n = 1/(2n)$ and the sequence of functions $\{v_n\}_{n\in\mathbb{N}}$ such that

$$v_n = w_{p_n} : [0, \infty) \to \mathbb{R}. \tag{57}$$

It now follows immediately from the Ascoli–Arzela Theorem (for details, see [13, Theorems 7.17 and 7.25]) that there exists a function $w_* : [0, \infty) \to \mathbb{R}$ such that $w_* \in C^1([0, \infty))$ and for any $X > 0$, the sequence of functions $\{v_n\}_{n\in\mathbb{N}}$ given by (57) has a subsequence $\{v_{n_l}\}_{l\in\mathbb{N}}$ ($1 \leq n_1 < n_2 < \ldots$ and $n_l \to \infty$ as $l \to \infty$) that satisfies

$$v_{n_l} \to w_* \text{ as } l \to \infty \text{ uniformly on } [0, X], \tag{58}$$

$$v_{n_l}' \to w_*' \text{ as } l \to \infty \text{ uniformly on } [0, X]. \tag{59}$$

Remark 4.5. Through Proposition 4.1, Proposition 4.2, (24), (57), (58) and (59), it follows that

$$0 \leq w_*(\eta) \leq 1, \quad 0 \leq w_*'(\eta) \leq \frac{2}{\sqrt{\pi}} \quad \forall \eta \geq 0,$$

and

$$w_*(0) = 0,$$

whilst via Proposition 4.3,

$$w_*(\eta) \geq \begin{cases} \frac{1}{8\sqrt{2}}\eta, & 0 \leq \eta \leq \eta' \\ \frac{\eta'}{8\sqrt{2}}, & \eta > \eta'. \end{cases}$$

We now have,

Proposition 4.6. *Let $X_2 > X_1 > 0$. Then $w_* \in C^2([X_1, X_2])$ and,*

$$w_*'' + \frac{1}{2}\eta w_*' + 1 - w_* = 0 \quad \forall \eta \in [X_1, X_2].$$

Proof. Set $X_2 > X_1 > 0$. It then follows from (58) and (59) that there is a subsequence $\{v_{n_l}\}_{l\in\mathbb{N}}$ of $\{v_n\}_{n\in\mathbb{N}}$ such that

$$v_{n_l} \to w_* \text{ as } l \to \infty \text{ uniformly on } [X_1, X_2], \tag{60}$$

$$v_{n_l}' \to w_*' \text{ as } l \to \infty \text{ uniformly on } [X_1, X_2]. \tag{61}$$

Also, via (57) and (21),

$$v_{n_l}'' = -\frac{1}{2}\eta v_{n_l}' - (v_{n_l})^{p_{n_l}} + \frac{v_{n_l}}{(1-p_{n_l})} \quad \forall \eta \in [X_1, X_2]. \tag{62}$$

We now observe that w_* is bounded above zero on $[X_1, X_2]$, via Remark 4.5, and so it follows from (60)–(62), that

$$v_{n_l}'' \to -\frac{1}{2}\eta w_*' - 1 + w_* \text{ as } l \to \infty \text{ uniformly on } [X_1, X_2]. \tag{63}$$

Finally, via (63) and [13, Theorem 7.17], we conclude that $w_* \in C^2([X_1, X_2])$ and

$$v_{n_l}'' \to w_*'' \text{ as } l \to \infty \text{ uniformly on } [X_1, X_2]. \tag{64}$$

The proof is completed via (63), (64) and the uniqueness of limits. □

We now investigate the behavior of $w_* : [0, \infty) \to \mathbb{R}$ as $\eta \to \infty$. To begin, we have,

Lemma 4.7. *Consider $w_p : [0, \infty) \to \mathbb{R}$ with $p \in (0, 1/2]$. Then,*

$$0 < w_p'(\eta) < \frac{2}{\sqrt{\pi}} e^{-\eta^2/4} \quad \forall \eta \geq 0, \tag{65}$$

and

$$-2\,\mathrm{erfc}\left(\frac{1}{2}\eta\right) \leq w_p(\eta) - (1-p)^{1/(1-p)} \leq 0 \quad \forall \eta \geq 0.$$

Proof. Via (21), (23), (25) and (26), we have

$$w_p'' + \frac{\eta}{2} w_p' = \frac{1}{(1-p)} w_p - f_p(w_p) < 0 \quad \forall \eta \in (0, \infty). \tag{66}$$

Therefore,

$$w_p'(\eta) < w_p'(0) e^{-\eta^2/4} \quad \forall \eta \geq 0. \tag{67}$$

The inequality in (65) then follows from (67) and Proposition 4.2. An integration of (65) then gives

$$0 < w_p(\eta_l) - w_p(\eta) < \frac{2}{\sqrt{\pi}} \int_{\eta}^{\eta_l} e^{-\lambda^2/4} d\lambda \tag{68}$$

for any $\eta_l > \eta > 0$. Allowing $\eta_l \to \infty$ in (68), using (24), then results in

$$-2\,\mathrm{erfc}\left(\frac{1}{2}\eta\right) \leq w_p(\eta) - (1-p)^{1/(1-p)} \leq 0 \quad \forall \eta \geq 0,$$

as required. □

We now have

Corollary 4.8. *Consider* $w_p : [0, \infty) \to \mathbb{R}$ *with* $p \in (0, 1/2]$*. Then,*

$$w_p(\eta) \to (1-p)^{1/(1-p)} \text{ as } \eta \to \infty \text{ uniformly for } p \in (0, 1/2].$$

Proof. This follows immediately from Lemma 4.7. □

As a consequence of Corollary 4.8, we now have

Lemma 4.9. *The function* $w_* : [0, \infty) \to \mathbb{R}$ *satisfies,*

$$w_*(\eta) \to 1 \text{ as } \eta \to \infty.$$

Proof. It follows from Remark 4.5 that

$$\limsup_{\eta \to \infty} w_*(\eta) \leq 1. \tag{69}$$

Now, from Corollary 4.8, for any $\epsilon > 0$, there exists $\eta^* > 0$ (dependent only upon ϵ) such that for all $p \in (0, 1/2]$, then

$$w_p(\eta) \geq (1-p)^{1/(1-p)} - \epsilon \quad \forall \eta \geq \eta^*. \tag{70}$$

Thus, via (70), (57) and (58),

$$w_*(\eta) \geq 1 - \epsilon \quad \forall \eta \geq \eta^*,$$

and so,

$$\liminf_{\eta \to \infty} w_*(\eta) \geq 1 - \epsilon. \tag{71}$$

Since (71) holds for any $\epsilon > 0$, then

$$\liminf_{\eta\to\infty} w_*(\eta) \geq 1. \tag{72}$$

It follows immediately from (69) and (72) that the limit of $w_*(\eta)$ as $\eta \to \infty$ exists and

$$\lim_{\eta\to\infty} w_*(\eta) = 1,$$

as required. □

We now have,

Proposition 4.10. *The function $w_* : [0, \infty) \to \mathbb{R}$ is given by*

$$w_*(\eta) = 1 - \frac{4(2+\eta^2)I(\eta)}{\sqrt{\pi}} \quad \forall \eta \geq 0.$$

Proof. It follows from (57)–(59) and Proposition 4.6 that $w_* \in C^1([0,\infty)) \cap C^2((0,\infty))$. Additionally, it follows from Proposition 4.6 that $w = w_*$ satisfies (40). Moreover, from Remark 4.5 and Lemma 4.9, it follows that $w = w_*$ satisfies (41) and (42). We thus conclude, that $w_* : [0,\infty) \to \mathbb{R}$ satisfies the boundary value problem (S_0). It has been established in §3 that (S_0) has a unique solution given by (43)–(44), as required. □

We immediately have,

Corollary 4.11. *There exists a subsequence $\{p_{n_l}\}_{l\in\mathbb{N}}$ of $\{p_n\}_{n\in\mathbb{N}}$ such that*

$$w'_{p_{n_l}}(0) \to \frac{2}{\sqrt{\pi}} \text{ as } l \to \infty.$$

Proof. It follows directly from (57), (58) and Remark 4.5 that there exists a subsequence $\{p_{n_l}\}_{l\in\mathbb{N}}$ of $\{p_n\}_{n\in\mathbb{N}}$ such that,

$$w'_{p_{n_l}}(0) \to w'_*(0) \text{ as } l \to \infty.$$

However, from Proposition 4.10, (43) and (45),

$$w'_*(0) = w'_0(0) = \frac{2}{\sqrt{\pi}}$$

and the proof is complete. □

We are now able to give the proof of our main result,

Proof of Theorem 1.3. First fix $\alpha > 0$, $T > 0$ and $N \in \mathbb{N}$. Next consider the subsequence $\{p_{n_l}\}_{l\in\mathbb{N}}$ of $\{p_n\}_{n\in\mathbb{N}}$ corresponding to that in Corollary 4.11. Let the constant

$$c(\alpha, T) = \frac{\sqrt{\pi}}{2T^{1/2}}(\alpha + 1). \tag{73}$$

We now introduce the sequence of functions $\{u^{(l)} : \bar{D}_T \to \mathbb{R}\}_{l\in\mathbb{N}}$ as

$$u^{(l)}(x,t) = \begin{cases} w_{p_{n_l}}\left(\frac{x_1}{\sqrt{t}}\right)(c(\alpha,T)t)^{1/(1-p_{n_l})}, & (x,t) \in [0,\infty)\times\mathbb{R}^{N-1}\times(0,T] \\ -w_{p_{n_l}}\left(\frac{-x_1}{\sqrt{t}}\right)(c(\alpha,T)t)^{1/(1-p_{n_l})}, & (x,t) \in (-\infty,0)\times\mathbb{R}^{N-1}\times(0,T] \\ 0, & (x,t)\in\partial D. \end{cases} \tag{74}$$

It is straightforward to verify directly, via (73), (74), (57), (20) and (27), that for each $l \in \mathbb{N}$,

$$(c(\alpha,T)f_{p_{n_l}}, 0, u^{(l)}) \in \mathcal{I}_t \quad \forall t \in (0,T]. \tag{75}$$

In addition, via (74), (20) and (25), we have

$$\max_{i=1,\ldots,N} ||u^{(l)}_{x_i}(\cdot,t)||_\infty = ||u^{(l)}_{x_1}(\cdot,t)||_\infty = w'_{p_{n_l}}(0)c(\alpha,T)^{1/(1-p_{n_l})}t^{(1+p_{n_l})/2(1-p_{n_l})} \quad \forall t\in(0,T], \tag{76}$$

and so, in particular

$$\max_{i=1,\ldots,N} ||u^{(l)}_{x_i}(\cdot,T)||_\infty \to (\alpha+1) \text{ as } l\to\infty, \tag{77}$$

via (76) and (73) with Corollary 4.11. Therefore, there exists $L\in\mathbb{N}$ such that

$$\max_{i=1,\ldots,N} ||u^{(l)}_{x_i}(\cdot,T)||_\infty \geq \alpha \quad \forall l \geq L. \tag{78}$$

Therefore, for each $l \geq L$,

$$(c(\alpha,T)f_{p_{n_l}}, 0, u^{(l)}) \in \mathcal{I}^\alpha_T. \tag{79}$$

Finally, it follows as in (36) that for each $l \geq L$,

$$\inf_{t\in(0,T]}\left(\max_{i=1,\ldots,N} ||u^{(l)}_{x_i}(\cdot,t)||_\infty - \mathcal{F}_t(c(\alpha,T)f_{p_{n_l}}, 0, u^{(l)})\right)$$
$$= c(\alpha,T)^{1/(1-p_{n_l})}T^{(1+p_{n_l})/2(1-p_{n_l})}\left(w'_{p_{n_l}}(0) - \phi(p_{n_l})\right)$$

and so, via Corollary 4.11 and (33),

$$\lim_{l\to\infty}\left(\inf_{t\in(0,T]}\left(\max_{i=1,\ldots,N} ||u^{(l)}_{x_i}(\cdot,t)||_\infty - \mathcal{F}_t(c(\alpha,T)f_{p_{n_l}}, 0, u^{(l)})\right)\right)$$
$$= c(\alpha,T)T^{1/2}\left(\frac{2}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}\right) = 0,$$

and the proof of Theorem 1.3 is complete. □

5. Discussion

We note here that it is not possible to establish a proof of Theorem 1.3 with a sequence of the form $\{(g_n, 0, u^{(n)}) \in \mathcal{I}_T\}_{n\in\mathbb{N}}$ with $g_n : \mathbb{R} \to \mathbb{R}$ anti-symmetric, Lipschitz continuous, and such that $g_n(u) \to 1$ as $n \to \infty$ for each $u > 0$. This follows since $u^{(n)} : \bar{D}_T \to \mathbb{R}$ is the unique solution to

$$u_t^{(n)} - \triangle u^{(n)} - g_n(u^{(n)}) = 0 \text{ on } D_T,$$

$$u^{(n)} = 0 \text{ on } \partial D,$$

and so $u^{(n)} = 0$ on $\bar{D}_T$ for each $n \in \mathbb{N}$, via the uniqueness of solutions (see, for example, [14, Theorem 4.5]). However, we anticipate that a proof of Theorem 1.3 may be established, somewhat more generically, by considering a sequence of the form $\{(g_n, u_0, u^{(n)}) \in \mathcal{I}_T\}_{n\in\mathbb{N}}$, with g_n and $u^{(n)}$ defined as above, but now, with non-zero initial data $u_0 : \mathbb{R}^N \to \mathbb{R}$ given by

$$u_0(x) = \begin{cases} w_0\left(\frac{x_1}{\sqrt{\lambda_0}}\right)\lambda_0, & x \in [0,\infty) \times \mathbb{R}^{N-1} \\ -w_0\left(\frac{-x_1}{\sqrt{\lambda_0}}\right)\lambda_0, & x \in (-\infty, 0) \times \mathbb{R}^{N-1}, \end{cases}$$

for some fixed $\lambda_0 > 0$, with $w_0 : [0, \infty) \to \mathbb{R}$ given by (43)–(44).

Alternatively, since [CP] is often stated (when considered in applications) with an additional decay condition as $|x| \to \infty$, for example,

$$u(x,t) \to 0 \text{ as } |x| \to \infty \text{ uniformly for } t \in [0, T], \tag{80}$$

one can inquire if the functional derivative estimate in Proposition 1.1 is non-trivially sharp when now, a solution to [CP] satisfies (1)–(3) and (80). Since w_p satisfying (21)–(26) can be continuously deformed into functions $\tilde{w}_p$ (for details, see [4]) which satisfy (21)–(23), and (25)–(26) for $w_p = \tilde{w}_p$, and

$$\lim_{\eta\to\pm\infty} \tilde{w}_p(\eta) = 0,$$

it follows that the associated analogue of Theorem 1.3 holds when $N = 1$, however, is open for $N \in \mathbb{N}\backslash\{1\}$. Additionally, we note that it is likely that results of similar type to Theorem 1.3 can be established for functional derivative estimates of the Dirichlet and Neumann problems associated with (1)–(3).

Acknowledgments

The first author was financially supported by an EPSRC PhD Prize. Additionally, the authors thank Prof. J. Bennett for reading an early version of the manuscript, and the reviewer, for helpful suggestions.

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